Intro to Topology: Midterm Review

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Suggested Use. Try to remember everything you can about each topic before reading these notes, then look over them and see what you've forgotten/aren't able to prove.

1 Basics

- Given a set X, a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* of X if the following conditions are satisfied:

  (a) $\emptyset, X \in \mathcal{T}$.

  (b) $\mathcal{T}$ is closed under (arbitrary) unions, i.e. for any $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup_{U\in\mathcal{S}} U \in \mathcal{T}$.

  (c) $\mathcal{T}$ is closed under *finite* intersections, i.e. for any *finite* subset $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{U\in\mathcal{S}} U \in \mathcal{T}$.

  The elements $U \in \mathcal{T}$ are called *open sets*, and for each $x \in U$ we say that U is a *neighborhood* of x. We will call the pair $(X, \mathcal{T})$ a *topological space*. When $\mathcal{T}$ is clear from context we simply write X instead of $(X, \mathcal{T})$.

- E.g. in the Euclidean topology on $\mathbb{R}^n$, a set U is open iff for all $x \in U$ there exists an open ball B containing x contained in U.

- E.g. if X is a space and $A \subseteq X$, then the subspace topology on A has $V \subseteq A$ open iff there exists an open set U in X such that $V = U \cap A$.

  – Proof that this is actually a topology: $\emptyset, X$ are easy. For finite intersections, if you have $V_1, \ldots, V_r$ open then $V_i = U_i \cap Y$ for some U_i, then $\bigcap V_i = \bigcap U_i \cap Y$ which is open since X is a topology. The proofs for unions is similar

- Given a space X, a set $C \subseteq X$ is said to be closed if $X - C$ is open.

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- For a space X and $A \subseteq X$, the closure of A, denoted $\overline{A}$, can be defined in many equivalent ways. The most useful definition tends to be that $x \in \overline{A}$ iff every neighborhood of x intersects A.

- Given a topological space X, we say that a sequence of points $(x_n)_{n\geq 1}$ in X *converges* to a point x if for all neighborhoods U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

- A topological space is said to be *Hausdorff* if for each pair of distinct points x, y, there exist neighborhoods U, V of x, y respectively which are disjoint.

- Important property 1 of Hausdorff spaces: if X is Hausdorff, then every sequence x_n converges to at most one point.

  Proof idea: if there were two limits x, y, take U, V containing each of them and disjoint, a sequence can't eventually end up in both of them.

- Important property 2 of Hausdorff spaces: if X is Hausdorff, then every one-point subset $\{x\} \subseteq X$ is closed (and hence every finite subset is closed).

  Proof idea: use the "neighborhood trick." Want to prove $X - \{x\}$ is open. For each $y \in X - \{x\}$, Hausdorff implies there exists open neighborhood $y \in V_y \subseteq X - \{x\}$, the union of these V_y equals $X - \{x\}$ and is open (since arbitrary unions of open sets are open).

2 Continuous Functions and Basis

- A function $f : X \to Y$ is continuous if "the preimage of open sets are open", i.e. for every $V \subseteq Y$ open the preimage $f^{-1}(V)$ is open in X.

- E.g. the composition of two continuous functions $f : X \to Y$ and $g : Y \to Z$ is continuous (because $(g \circ f)^{-1}(U) = g^{-1}(f^{-1}(U))$).

- A map $f : X \to Y$ is said to be a *homeomorhism* if (a) f is a bijection, (b) f is continuous, and (c) f^{-1} is continuous (or equivalently, $f(U)$ is open whenever U is open).

  If there exists a homeomorphism between X, Y we say these spaces are *homeomorphic* and write $X \cong Y$.

- E.g. the intervals $(0, 1)$ and $(1, 2)$ are homeomorphic (since we know translation map $f(x) = x + 1$ is continuous by calculus).

- Given a set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* of X if (1) for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x and (2) for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.

  E.g. open balls in $\mathbb{R}^n$ is a basis (proof by picture).

- If $\mathcal{B}$ is a basis for X, the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ has two equivalent definitions:

  First definition (less useful): $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

  Second definition (more useful): $U \in \mathcal{T}$ if and only if U can be written as the union of elements of $\mathcal{B}$.

- Important property 1 of basis: if Y is generated by a basis $\mathcal{B}$, then $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

- Important property 1 of basis: if X is generated by a basis $\mathcal{B}$ and $A \subseteq X$, then $x \in \overline{A}$ iff every $B \in \mathcal{B}$ containing x intersects A.

3 Products

- Given topological spaces X, Y, define the *product topology* on $X \times Y$ as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ open in } X, V \text{ open in } Y\}$.

  Warning: not all open sets in $X \times Y$ are of the form $U \times V$ (e.g. the union of two rectangles is open.

- Main property of product topology: a function $f : Z \to X_1 \times X_2$ of the form $f(z) = (f_1(z), f_2(z))$ is continuous iff each of the maps $f_i : Z \to X_i$ are continuous.

- Recap of set theoretic product notation (skip if no one wants to see this and/or if low on time).

  - Given sets J, X, we define a J-tuple of elements of X to be a function $\mathbf{x} : J \to X$. If $\alpha \in J$ we often denote the value $\mathbf{x}(\alpha)$ by x_α and denote $\mathbf{x}$ by the symbol $(x_\alpha)_{\alpha \in J}$.

  - Given an indexed family of sets $\{A_\alpha\}_{\alpha \in J}$, we define the cartesian product $\prod_{\alpha \in J} A_\alpha$ to be the set of all J-tuples of $X = \bigcup A_\alpha$ such that $x_\alpha \in A_\alpha$ for all $\alpha \in J$.

    If $A_\alpha = X$ for all $\alpha \in J$, then we will write this product as X^J (equivalent to set of all functions from J to X), and if $J = \mathbb{Z}_{>0}$ we use the shorthand X^ω.

  - Eg if $J = \{1, 2\}$ then $A_1 \times A_2$ consists of all functions $x : \{1, 2\} \to A_1 \cup A_2$ with $x_1 \in A_1$ and $x_2 \in A_2$. This is just the usual definition.

  - Eg if $J = \mathbb{Z}_{>0}$ and $A_\alpha = \mathbb{R}$ for all α what is $\prod A_\alpha = \mathbb{R}^{\mathbb{Z}_{>0}} = \mathbb{R}^\omega$? Formally this is all functions $f : \mathbb{Z}_{>0} \to \mathbb{R}$, which (in tuple notation) is the set of sequences of real numbers (e.g. $(n^2)_{n \geq 1}$ is in this set).

- Given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$, we define the *box topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open.

  This is not so useful.

- Given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$, we define the *product topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open **and** where $U_\alpha = X_\alpha$ for all but finitely many α.

  This is useful.

- Main property of product topology: let $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ be the projection map onto the β coordinate. If $\prod X_\alpha$ has product topology then a map $f : Z \to \prod X_\alpha$ is continuous iff $\pi_\alpha \circ f$ is continuous for all α.

- Other important property of products: a sequence of points $x_n \in \prod X_\alpha$ under the product topology converges to a point x iff $(x_n)_\alpha$ converges to x_α for all α (i.e. the product topology is the topology of pointwise convergence).

4 Metric Spaces

- Given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* if (1) $d(x, y) \geq 0$ for all $x, y \in X$ with equality iff $x = y$, (2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and (3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

- Given a metric d, a point $x \in X$, and a real number $\varepsilon > 0$, we define the open ball $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

- Given a metric d, the metric topology induced by d is the topology generated by the basis of open balls $B_d(x, \varepsilon)$. Note that U is open iff for every $x \in U$ we have some $B_d(x, \varepsilon) \subseteq U$.

- We say that a topological space X is metrizable if there exists a metric d on X which induces the topology of X. A pair (X, d) is a metric space if X is a metrizable topology and d is a metric inducing the topology on X.

- Important property of metrizable spaces 1: if X is metrizable, then X is Hausdorff.

  Proof idea: after specifying a metric d, we can take neighborhoods for x, y to be the balls of radius $d(x, y)/2$ centered at each of x, y.

- Important property of metrizable spaces 1: (Sequence lemma) Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.

  Proof idea: first part is easy unwinding of definition. For second part, after specifying a metric we take x_n to be a point in A intersecting $B_d(x, 1/n)$ (exists because $x \in \overline{A}$); this sequence turns out to converge to x.

- Let $f_n : X \to Y$ be a sequence of functions with (Y, d) a metric space. We say that (f_n) *converges uniformly* to a function $f : X \to Y$ if for all $\varepsilon > 0$ there exists an N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$ and all $x \in X$.

- Uniform limit theorem: let $f_n : X \to Y$ be a sequence of continuous functions with Y a metric space. If (f_n) converges uniformly to f, then f is continuous.

5 Quotient Spaces

They only need to know how to intuitively identify quotient spaces. As such you might draw e.g. the quotient diagram for a torus (a square with opposite sides oriented in the same way) and explain how to do this. Can ask them for other spaces from notes/homework that they want clarification on (if any).